Space Time Foam: a ground state candidate for Quantum Gravity

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A model of space-time foam, made by N wormholes is considered. The Casimir energy leading to such a model is computed by means of the phase shift method which is in agreement with the variational approach used in Refs. [9–14]. The collection of Schwarzschild and Reissner-Nordström wormholes are separately considered to represent the foam. The Casimir energy shows that the Reissner-Nordström wormholes cannot be used to represent the foam.

I. INTRODUCTION

A very crucial question induced by the appearance of quantum phenomena at the Planck scale is: what happens when the metric fluctuations become large? One possible answer should be extracted from the traditional path integral approach to quantum gravity

$$\int \mathcal{D}g_{\mu\nu} \exp iS_g\left[g_{\mu\nu}\right]. \tag{1}$$

Unfortunately this quantity is ill defined because the symbol $[\mathcal{D}g_{\mu\nu}]$ does not represent a measure. However in the context of the background field method, a WKB method gives interesting results. In this context, we can approximate Eq.(1) with

$$\Gamma = A \exp\left(-I_{cl}\right), \tag{2}$$

where A is the prefactor coming from the saddle point evaluation and I_{cl} is the classical part of the Euclidean action. If a single negative eigenvalue appears in the prefactor A,it means that the related bounce shifts the energy of the false ground state [1]. In particular in this approximation, it is possible to discuss decay probabilities from one space-time to another one [2–7]. For a certain class of gravitational backgrounds, namely the static spherically symmetric metrics, it could be interesting the use of other methods based on variational approach. In a series of papers, we have used such an approach to show that a model of space-time foam [8] can be concretely realized if one considers a collection of Schwarzschild wormholes whose energy is given by Casimir energy [9–14]. We recall that the Casimir energy procedure involves a subtraction procedure between zero point energies having the same boundary conditions. In this paper we compare the variational approach with the more traditional phase shift representation of the Casimir energy. We will consider two types of static spherically symmetric wormholes: the Schwarzschild wormhole and the Reissner-Nordström (RN) wormhole. The final energies will be compared showing that the Casimir energy for RN wormholes is always higher than the Casimir energy for the Schwarzschild wormholes. This means that RN wormholes cannot be taken as a representation of a ground state of a foamy space-time. To this purpose we will fix our attention on the following quantity

$$E\left(wormhole\right) = E\left(no-wormhole\right)$$

$$+ \Delta E^{wormhole}_{no-wormhole|classical} + \Delta E^{wormhole}_{no-wormhole|1-loop}, \tag{3}$$

representing the total energy computed to one-loop in a wormhole background. E (no wormhole) is the reference space energy which, in the case of the Schwarzschild and RN wormhole, is flat space. $\Delta E^{wormhole}_{no-wormhole|classical}$ is the classical energy difference between the wormhole and no-wormhole configuration stored in the boundaries and finally $\Delta E^{wormhole}_{no-wormhole|1-loop}$ is the quantum correction to the classical term. It is possible to proof that in the considered foam model, the classical term is always vanishing [9–14]. The one-loop contribution can be composed, other than by a stable spectrum, even by an unstable spectrum. If the unstable spectrum is composed by exactly one element, we can invoke Coleman arguments to conclude that we move from a false vacuum towards the true one like in the Euclidean path integral formulation. Nevertheless in the foam model, we know that the instability can be eliminated in the large N_w-wormhole approach as a wormhole packing consequence [11–13]. Thus to compare two different wormhole models of foam, it is sufficient to assume the existence of an unstable mode, which will be subsequently eliminated and compare only the stable spectrum . In particular, it is the following inequality

$$\Delta E_{flat}^{RN}\left(M,Q\right) \lessgtr \Delta E_{flat}^{Schwarzschild}\left(M\right) \tag{4}$$

that will be taken under examination. The final result will give indications on the possible *"ground state"* of a foamy space-time. Inequality (4) can be examined by means of Casimir energy, which in the variational language will be expressed by the following expectation value [9–13]

$$\Delta E_{no-wormhole|1-loop}^{wormhole} = \frac{\left\langle \Psi \left| H_\Sigma^{wormhole} - H_\Sigma^{no-wormhole} \right| \Psi \right\rangle}{\left\langle \Psi | \Psi \right\rangle}. \tag{5}$$

where Ψ is a *trial wave functional* of the gaussian form. The computation of Eq.(5) can be easily generalized to a system of N_w wormholes such that the hypersurface Σ is such that $\Sigma = \bigcup_{i=1}^{N_w} \Sigma_i$, with $\Sigma_i \cap \Sigma_j = \emptyset$ when $i \neq j$. Thus the total energy will be simply

$$E^{foam} = N_w \Delta E_{no-wormhole|1-loop}^{wormhole}. \tag{6}$$

This N_w wormholes system will be considered as a model for space-time foam (Units in which $\hbar = c = k = 1$ are used throughout the paper).

II. THE WORMHOLE METRIC AND THE ENERGY OF THE FOAM

The line element we consider is

$$ds^2 = -N^2\left(r\right) dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{7}$$

where $N\left(r\right)$ is the lapse function and $b\left(r\right)$ is the shape function such that[1]

$$b\left(r\right) = \begin{cases} 2MG & \text{Schwarzschild} \\ 2MG - Q^2/r & \text{Reissner-Nordström} \end{cases}. \tag{8}$$

M is the wormhole mass, while $Q^2 = G\left(Q_e^2 + Q_m^2\right)$; Q_e and Q_m are the electric and magnetic charge respectively. The wormhole throat r_h is located at

$$r_h = \begin{cases} 2MG & \text{Schwarzschild} \\ r_+ = MG + \sqrt{\left(MG\right)^2 - Q^2} & \\ r_- = MG - \sqrt{\left(MG\right)^2 - Q^2} & \text{Reissner-Nordström} \end{cases}. \tag{9}$$

When $Q = 0$ the metric describes the Schwarzschild metric. When $Q = M = 0$, the metric is flat. For $Q \neq 0$, we shall consider only the case $MG > Q$. In a W.K.B. approximation, Eq.(5) can be easily computed. If we restrict to the physical sector of TT (transverse-traceless) tensors, the Hamiltonian is approximated by

$$H^\perp = \frac{1}{4} \int_\Sigma d^3x \sqrt{g} G^{ijkl} \left[\left(16\pi G\right) K^{-1\perp}\left(x,x\right)_{ijkl} + \frac{1}{16\pi G} \left(\triangle_2\right)_j^a K^\perp\left(x,x\right)_{iakl} \right], \tag{10}$$

where we have considered on Σ perturbations of the form

$$g_{ij} = \bar{g}_{ij} + h_{ij}, \tag{11}$$

with $\bar{g}_{ij}$ corresponding to the spatial part of the metric of Eq.(7). The propagator $K^\perp\left(x,x\right)_{iakl}$ comes from a functional integration and it can be represented as

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[1]Nothing prevents to consider a positive or negative cosmological constant in the metric. However, in this paper, the discussion will be restricted to charged and neutral wormholes.

$$K^\perp \left(\overrightarrow{x}, \overrightarrow{y} \right)_{iakl} := \sum_\tau \frac{h_{ia}^{(\tau)\perp} \left(\overrightarrow{x} \right) h_{kl}^{(\tau)\perp} \left(\overrightarrow{y} \right)}{2\lambda \left(\tau \right)}, \tag{12}$$

where $h_{ia}^{(\tau)\perp} \left(\overrightarrow{x} \right)$ are the eigenfunctions of $\triangle_2$. τ denotes a complete set of indices and $\lambda \left(\tau \right)$ are a set of variational parameters to be determined by the minimization of Eq.(10). The expectation value of $H^\perp$ is easily obtained by inserting the form of the propagator into Eq.(10)

$$E \left(M, Q, \lambda_i \right) = \frac{1}{4} \sum_\tau \sum_{i=1}^2 \left[\left(16\pi G \right) \lambda_i \left(\tau \right) + \frac{E_i^2 \left(\tau \right)}{\left(16\pi G \right) \lambda_i \left(\tau \right)} \right], \tag{13}$$

where we have pointed out the dependence of the energy on some parameters like the mass and charge. By minimizing with respect to the variational function $\lambda_i \left(\tau \right)$ we get

$$E \left(M, Q \right) = \frac{1}{2} \sum_\tau \left[\sqrt{E_1^2 \left(\tau \right)} + \sqrt{E_2^2 \left(\tau \right)} \right]. \tag{14}$$

The above expression makes sense only for $E_i^2 \left(\tau \right) > 0$, $i = 1, 2$. To complete Eq.(5), we have to subtract the zero point energy contribution of the space without wormhole: this is the Casimir energy generated by the curvature potential. In terms of phase shifts, the Casimir energy is

$$\frac{1}{2} \int_0^{+\infty} dp\, p \sum_{l=0}^{+\infty} \left[\rho_l \left(p \right) - \rho_l^{(0)} \left(p \right) \right]^\pm = \frac{1}{2\pi} \int_0^{+\infty} dp\, p \sum_{l=0}^{+\infty} \left(2l + 1 \right) \frac{\partial}{\partial p} \delta_l^\pm \left(p \right), \tag{15}$$

where $\rho_l \left(p \right)$ represents the density of states in wormhole background ($\rho_l^{(0)} \left(p \right)$ represents the density of states in absence of the wormhole, respectively) and $\delta_l^\pm \left(p \right)$ is the phase shift due to the curvature potential. Thus the total Casimir energy is

$$\Delta E = E^{wormhole} - E^{no-wormhole}$$

$$= \frac{1}{2\pi} \int_0^{+\infty} dp \int_0^{+\infty} dl \left(2l + 1 \right) \left[\left(\frac{d\delta_l^+ \left(p \right)}{dp} + \frac{d\delta_l^- \left(p \right)}{dp} \right) \right] p, \tag{16}$$

where we have replaced the sum with an integration over all modes. The phase shift is defined as $\left(r \equiv r \left(x \right) \right)$

$$\delta_l^\pm \left(p \right) = \lim_{R \to +\infty} \left[\int_{r_h}^{x(R)} dx \sqrt{p^2 - \frac{l \left(l + 1 \right)}{r^2} - \tilde{V}^\mp \left(r \right)} - \int_{r_h}^{x(R)} dx \sqrt{p^2 - \frac{l \left(l + 1 \right)}{r^2}} \right], \tag{17}$$

where x is the proper distance from the throat and $\tilde{V}^\mp \left(r \right)$ is the curvature potential due to the wormhole. If we first integrate over the angular momenta with the condition that the square root be real and then we integrate over p with the condition $p \geq \tilde{V}^\mp \left(r \right)$, we get

$$\Delta E = \frac{1}{2\pi} \int_0^{+\infty} dp \int_0^{+\infty} dl \left(2l + 1 \right) \left[\left(\frac{d\delta_l^+ \left(p \right)}{dp} + \frac{d\delta_l^- \left(p \right)}{dp} \right) \right] p$$

$$= \frac{1}{\pi} \lim_{R \to +\infty} \int_{x(r_h)}^{x(R)} dx\, r^2 \int_0^{+\infty} dp\, p^2 \left[\sqrt{p^2 - \tilde{V}^+ \left(r \right)} + \sqrt{p^2 - \tilde{V}^- \left(r \right)} - 2p \right]$$

$$= \frac{V}{4\pi^2} \left[\Lambda^2 \frac{\tilde{V}^+ \left(r_0 \right) + \tilde{V}^- \left(r_0 \right)}{4} \right.$$

$$\left. - \left(\frac{\tilde{V}^+ \left(r_0 \right)}{4} \right)^2 \ln \left(\frac{\Lambda^2}{\left(\tilde{V}^+ \left(r_0 \right) \right)^2} \right) - \left(\frac{\tilde{V}^- \left(r_0 \right)}{4} \right)^2 \ln \left(\frac{\Lambda^2}{\left(\tilde{V}^- \left(r_0 \right) \right)^2} \right) \right], \tag{18}$$

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where we have introduced a cut-off Λ to keep under control the U.V. divergence, a radius r_0 such that $r_0 > r_h$ with $r_0 \neq \alpha r_h$. α is a constant and V is a *"local"* volume defined by

$$V = 4\pi \int_{x(r_h)}^{x(r_0)} dx r^2. \tag{19}$$

A comment to justify the approximation leading to Eq.(18) can be useful: since $\tilde{V}^{\mp}(r) = O\left(1/r^3\right)$, i.e. it is a short range curvature potential and since we are probing Planckian energies, the contribution to ΔE comes from the region close to the throat. One could be tempted to set $r_0 = r_h$. However a peculiar situation manifests in this limit. Indeed

$$\lim_{M \to 0} \lim_{r \to r_h} \Delta E(M, Q) \neq \lim_{r \to r_h} \lim_{M \to 0} \Delta E(M, Q). \tag{20}$$

One possible interpretation of this fact is related to the fluctuation of the throat enforcing therefore the choice $r_0 > r_h$. The form of ΔE changes from a case to case. Here we consider:

1. the Schwarzschild wormhole characterized by one parameter: the wormhole mass M.

$$\tilde{V}^{+}(r_0) = \frac{3MG}{r_0^3}, \qquad \tilde{V}^{-}(r_0) = -\frac{3MG}{r_0^3}, \tag{21}$$

$$\Delta E \equiv \Delta E(M) = -\frac{V}{32\pi^2}\left[\left(\frac{3MG}{r_0^3}\right)^2 \ln\left(\frac{\Lambda^2}{(3MG/r_0^3)}\right)\right], \tag{22}$$

   which is in complete agreement with variational approach used in Refs. [9–14].

2. The Reissner-Nordström wormhole characterized by two parameters: the wormhole mass M and the charge Q with $Q^2 = G\left(Q_e^2 + Q_m^2\right)$; Q_e and Q_m are the electric and magnetic charge respectively.

   a) electric charge Q_e

$$\tilde{V}^{+}(r_0) = \frac{3MG}{r_0^3} - \frac{3Q_e^2}{r_0^4}, \qquad \tilde{V}^{-}(r_0) = -\frac{3MG}{r_0^3} + \frac{9Q_e^2}{r_0^4},$$

$$\Delta E \equiv \Delta E(M, Q_e) = \frac{V}{4\pi^2}\left[\Lambda^2 \frac{3Q_e^2}{2r_0^4}\right.$$

$$\left. - \left(\frac{\tilde{V}^{+}(r_0)}{4}\right)^2 \ln\left(\frac{\Lambda^2}{\tilde{V}^{+}(r_0)}\right) - \left(\frac{\tilde{V}^{-}(r_0)}{4}\right)^2 \ln\left(\frac{\Lambda^2}{\tilde{V}^{-}(r_0)}\right)\right]. \tag{23}$$

   b) magnetic charge Q_m

$$\tilde{V}^{+}(r_0) = \frac{3MG}{r_0^3} + \frac{9Q_m^2}{r_0^4}, \qquad \tilde{V}^{-}(r_0) = -\frac{3MG}{r_0^3} + \frac{Q_m^2}{r_0^4},$$

$$\Delta E \equiv \Delta E(M, Q_m) = \frac{V}{4\pi^2}\left[\Lambda^2 \frac{5Q_m^2}{2r_0^4}\right.$$

$$\left. - \left(\frac{\tilde{V}^{+}(r_0)}{4}\right)^2 \ln\left(\frac{\Lambda^2}{\tilde{V}^{+}(r_0)}\right) - \left(\frac{\tilde{V}^{-}(r_0)}{4}\right)^2 \ln\left(\frac{\Lambda^2}{\tilde{V}^{-}(r_0)}\right)\right]. \tag{24}$$

   It is immediate to see that the presence of electric and magnetic charge, respectively gives a positive contribution to the difference of zero point energies. The final result is displayed in the following plots, where we have introduced a scale x such that $x = 3MG/r_0^3\Lambda^2$, a parameter α_e^2 with $0 < \alpha_e^2 < 1$ for the electric charge and a parameter α_m^2 with $0 < \alpha_m^2 < 1$ for the magnetic charge.
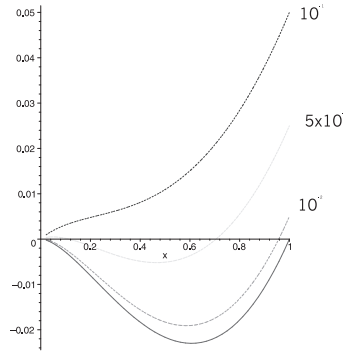
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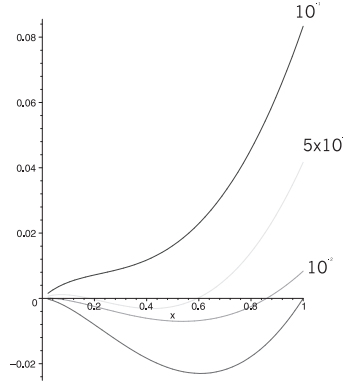FIG. 1. Plot of Electric charge contribution for different values of the parameter α_e^2.

FIG. 2. Plot of Magnetic charge contribution for different values of the parameter α_m^2.

The lowest curve corresponds to the value $\alpha_e^2 = \alpha_m^2 = 0$, namely the Schwarzschild case.

III. SUMMARY AND CONCLUSIONS

In this paper we have compared the variational approach to compute Casimir energy with the more traditional phase shift method: the two methods are in perfect agreement. Moreover we have considered a more general class of wormholes which includes a charge. This is the RN wormhole class. The examination of the Casimir energy shows that a space-time foam formation realized by RN wormholes is suppressed when compared with the foamy space formed by Schwarzschild wormholes. However, it is an open question the solution of the problem of a foamy space-time formed by a collection of N extreme RN wormholes. On the other hand one can think to the collection of N RN wormholes as an excited state with respect to the collection of N Schwarzschild wormholes leading to the conclusion that such a collection can be considered as a good candidate for a possible ground state of a quantum theory of the gravitational field, when compared to a superposition of large N RN wormholes.

IV. ACKNOWLEDGMENTS

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